Doeblin Graph of the null recurrent MC and Dynamics on it

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1 Abstract

Consider a Markov Chain X with countable state space $\mathcal{S}$. One can consider a natural dynamics, h, on the distributions on $\mathcal{S}$ related to this Markov Chain. When the Markov chain X is irreducible, aperiodic, and positive recurrent, h has a steady state (stationary) solution. And when the Markov Chain is null recurrent, h does not have a steady state solution. But it is known that a null recurrent Markov Chain admits a stationary measure. This lead us to consider another dynamics, related to the Markov Chain X, that admits the steady space in the null recurrent case. We define these dynamics on the random measures on $\mathcal{S}$. For study of the properties of these dynamics, we use the Doeblin graph. A bi-product of studying this random graph is that it gives us a way for defining a perfect sample of invariant measure of a null recurrent Markov Chain.

Key Words: Steady State, Dynamical System, Null Recurrent Markov Chain, Doeblin Graph, Bridge Graph, Unimodular random Graphs, Perfect Sampling

2 Summary and the Main Results

Let $\{X_t\}_{t\in\mathbb{Z}}$ be a Markov Chain with countable state space, $\mathcal{S}$. One can consider a natural dynamics on the distributions on $\mathcal{S}$ related to this MC. This dynamic can be written as a stochastic recurrence equation as follows

$$X_{t+1} = h(X_t, \eta(t)), \tag{1}$$

where $\{\eta(t)\}_{t\in\mathbb{Z}}$ is the source of randomness. When the Markov chain $\{X_t\}_{t\in\mathbb{Z}}$ is irreducible, aperiodic, and positive recurrent, the recurrence equation (1) has a steady state (stationary) solution. It means that there is a random variable X, with distribution π, such that $X \stackrel{d}{=} h(X, \eta(t))$, with $\stackrel{d}{=}$ meaning equality in distribution. When the Markov Chain is null recurrent the Equation (1), does not have a steady state solution. But the null recurrent Markov Chain admits stationary measure π, i.e, the measure π satisfies $\pi = \pi P$, where P is the transition matrix of the Markov Chain. This lead us to consider another dynamics relate to the Markov Chain $\{X_t\}_{t\in\mathbb{Z}}$, that admits the steady space in more general cases. We define these dynamics on the random measures on the state state space of the Markov Chain.

We leverage the Doeblin graph and the Bridge Graph of the Markov Chain to define appropriate dynamics and consider their properties.

The Doeblin graph of a Markov chain $\{X_t\}$, is a random graph with vertices in $\mathbb{Z} \times \mathcal{S}$. The x-axis in this graph represents time and the y-axis represents the state space. The edges of the Doeblin graph are defined using the transition probabilities of the Markov Chain. There is an edge from each vertex (t, x) to vertex $(t + 1, h(x, \xi(t))$. $\{\xi(t)\}_{t\in\mathbb{Z}}$ are independent sources of randomness for each t and $P(h(x, \xi(t)) = y)$ is equal to the transition probability from state x to y, in $\{X_t\}_{t\in\mathbb{N}}$. We consider our dynamics on the Doeblin graph. And using the properties of the Doeblin graph to understand the behavior of the Dynamics.

In the main part of the paper we restrict our attention to a subgraph of the Doeblin graph called the Bridge Graph. The Bridge Graph is the union of all paths of the Doeblin graph starting from $\mathbb{Z} \times \{x^*\}$, where x^* is an arbitrary point in $\mathcal{S}$. In the positive recurrent case, some properties of the Bridge Graph is known. It is shown in **??** that the Bridge Graph of a positive recurrent Markov Chain is a.s. connected (is a tree), locally finite, and unimodularisable. Using the unimodular property it is shown that in this case, there is a unique bi-infinite path in the Bridge graph. The distribution of the points in this bi-infinite path is the stationary distribution.

The first step of our work is to study the properties of the Bridge Graph in the case it is constructed from a null recurrent Markov Chain. In this case, we will show that the Bridge graph is not unimodularizable. And it can be both, a tree or a forest. And also we show that there is no bi-infinite path in the bridge graph in this case. We state them as the following propositions:

Proposition 2.1. *The Bridge graph constructed by a general null recurrent Markov Chain has no bi-infinite path.*

And about the unimodularity:

Proposition 2.2. *The Bridge graph constructed by a general null recurrent Markov Chain is not unimodularizable.*

Then we consider two main dynamics on the random measures. In the Bridge Graph, these dynamics are defined between the random measures that lie into two consequence times. The first dynamics is the Taboo dynamics. Using the properties of the Bridge Graph we will show that this dynamic has a steady state on the space of the random measures on $\mathcal{S}$, called the Taboo Point Process (TPP). The TPP has this property that the expectation of it at each point $x \in \mathcal{S}$ is equal to the stationary measure of the MC $\pi(x)$. We state it in the following theorem:

Theorem 2.3. *Consider a recurrent Markov Chain $\{X_n\}$ and its associated Bridge Graph B_X. The expectation of the PIR point process at each point in the S-set is equal to the stationary measure of that point in the Markov Chain $\{X_n\}$. That is,*

$$\mathbb{E}[\mu_{(0)}(i)] = \pi(i), \tag{2}$$

where π is the invariant measure of the Markov Chain $\{X_n\}$ and $\pi(0) = 1$.

Using this property, we consider a realization of TPP as a perfect sample of the stationary measure of the Markov Chain.
So we can summarise our result in this way: In comparison with Bridge Graph in the null recurrent case and the Bridge Graph in the Positive recurrent case, we have that this graph is not connected anymore in the null recurrent case. This graph is not unimodularisable in general, but it contains a unimodular graph. It does not have any bi-infinite path. Moreover, finally, we can have an analogy to perfect sampling in the null recurrent case, the Taboo PP.

3 Affiliation

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4 Dissermentation and Outputs

a. Talk A contributed talk in the 9th conference of Stochastic Geometry Days 2021 Date and palce: Dunkerque, November 15th - 19th, 2021

b. One finalized mathematical article. (near to submiting)

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