Surprising identities for the greedy independent set on Cayley trees

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Résumé. Nous prouvons que la taille G_n du greedy independent set sur un arbre de Cayley uniforme a (presque) la même loi que la taille de son complémentaire. En particulier, nous montrerons que G_n a la même loi que le nombre de sommets à hauteur paire dans un arbre de Cayley, ce qui nous permet de calculer la loi exacte de G_n. Nous montrerons aussi comment explorer G_n de façon markovienne.

Mots-clés. Arbres de Cayley, independent set.

Abstract. We prove a surprising symmetry between the law of the size G_n of the greedy independent set on a uniform Cayley tree $\mathcal{T}_n$ of size n and that of its complement. In particular, we show that G_n has the same law as the number of vertices at even height in $\mathcal{T}_n$ rooted at a uniform vertex. This enables us to compute the exact law of G_n. We also give a Markovian construction of the greedy independent set, which highlights the symmetry of G_n and whose proof uses a new Markovian exploration of *rooted* Cayley trees which is of independent interest.

Keywords. Cayley trees, independent set

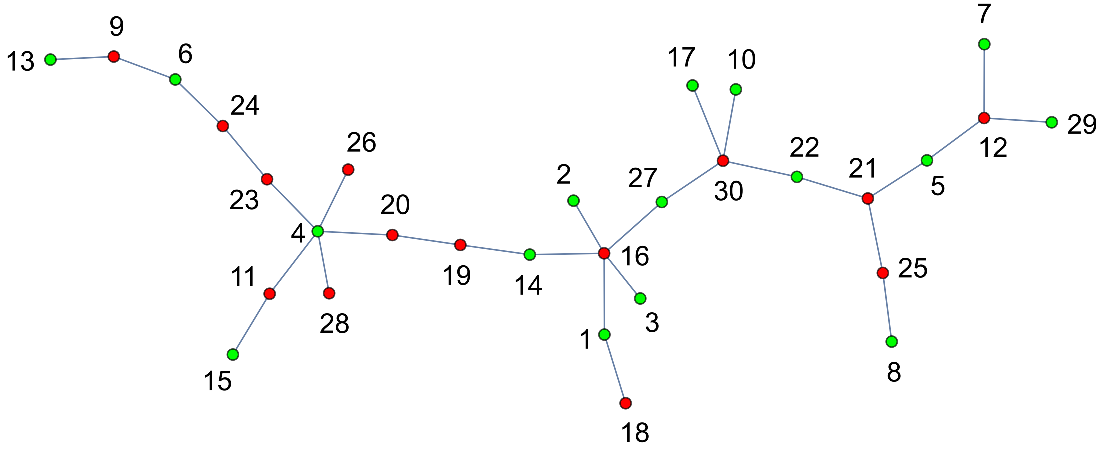

Figure 1: Example of the greedy independent set obtained on a tree of size 30. The labels represent the order in which vertices are inspected in the construction of the greedy independent set. The green vertices are the active vertices whereas the red vertices are the blocked vertices.

1 Introduction

An *independent* set of a graph $\mathcal{G} = (V, E)$ is a subset of vertices where no pair of vertices are connected to each other. Finding an independent set of maximal size is a notoriously difficult problem in general. However, using a greedy procedure, we can construct a *maximal* (for the inclusion order) independent set by inspecting the vertices of the graph one by one in a random order, adding the current vertex and blocking its neighbours if it is not connected to any previously added vertex. More precisely, the vertices are divided in three possible statuses: the undetermined vertices $\mathcal{U}_k$, the active vertices $\mathcal{A}_k$ and the blocked vertices $\mathcal{B}_k$. Initially, we start with $\mathcal{U}_0 = V$ and $\mathcal{A}_0 = \mathcal{B}_0 = \emptyset$. At step $k \geq 1$, we choose an undetermined vertex v_k uniformly at random, change its status to active and change the status of all its undetermined neighbours to blocked. We stop at $\tau = \min\{k \geq 0,\ \mathcal{U}_k = \emptyset\}$. Note that at each step k, no vertices of $\mathcal{A}_k$ are neighbours and $\mathcal{A}_\tau$ is a maximal independent set, which we call the (random) *greedy* independent set, see Figure 1.

Of course the independent set obtained by the greedy algorithm is usually not *maximum* in the sense that it does not have the maximal possible size. In the case of trees, finding an independent set of maximal size is much simpler than in general. However, from a probabilistic or combinatorial point of view the greedy independent set is still worth investigation even on (random) trees. Greedy independent sets on (random) graphs have been studied extensively with a particular focus on the proportion of vertices of the graph in the greedy independent set called the *greedy independence ratio* or *jamming constant.*

2 Results

In this talk, we will focus on Cayley trees. Recall that a *Cayley tree* of size n is an unrooted and unordered tree over the n labeled vertices $\{1, \ldots, n\}$ and we let $\mathcal{T}_n$ be a random Cayley tree sampled uniformly at random among the n^{n-2} Cayley trees of size n. We shall denote by $\mathcal{T}_n^\bullet$ the rooted tree obtained from $\mathcal{T}_n$ by distinguishing a vertex uniformly at random. Using the local limit of $\mathcal{T}_n^\bullet$ given by Kesten's infinite tree, Krivelevich, Mészáros, Michaeli and Shikelman (2020) proved the "intriguing fact" that the asymptotic greedy independence ratio of uniform Cayley trees is 1/2. Meir and Moon (1973) proved that the size of a maximum independent set of a uniform Cayley tree concentrates around ρn where $\rho \approx 0.5671$ is the unique solution of $xe^x = 1$.

In this talk, we present a much stronger, and perhaps surprising statement concerning the size of the greedy independent set on a uniform Cayley tree showing that it has (almost) the same law as that of its complement! We denote by G_n the size of the greedy independent set $|\mathcal{A}_\tau|$ on a uniform Cayley tree $\mathcal{T}_n$ and H_n the number of vertices at even height in $\mathcal{T}_n^\bullet$. Our first observation is that G_n has the same law as H_n, which enables us to compute the exact law of G_n.

Theorem 1 (C., 2021). The size G_n of the greedy independent set on $\mathcal{T}_n$ has the same law as the number H_n of vertices at even height in $\mathcal{T}_n^{\bullet}$. As a consequence, for $1 \leq k \leq n-1$,

$$\mathbb{P}(G_n = k) = \mathbb{P}(H_n = k) = \binom{n}{k} \frac{k^{n-k}(n-k)^{k-1}}{n^{n-1}}. \tag{1}$$

The proof of Theorem 1 relies on the invariance of Cayley trees under rerooting at a uniform vertex. The exact computation of the law of H_n is a consequence of a result of Féray and Kortchemski (2018) on bi-type alternating Galton–Watson trees. This equality in distribution of G_n and H_n suggests that their common law is almost symmetric with respect to $n/2$ with a little drift caused by the root vertex.

Theorem 2 (C., 2021). There exists a random variable $\mathcal{E}_n$ with values in $\{0,1\}$ such that we have

$$G_n \overset{(d)}{=} (n - G_n) + \mathcal{E}_n.$$

Moreover $\mathbb{P}(\mathcal{E}_n = 1) \to 1/4$ as n goes to ∞.

This symmetry between G_n and $n - G_n$ is striking because the geometry of a greedy independent set and that of its complement are totally different (see Figure 1).

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